q-Thermostatistics and the black-body radiation problem

S. Martínez [*], F. Pennini[†], A. Plastino[‡], and C. J. Tessone[§].

Instituto de Física de La Plata,

National University La Plata (UNLP) and National Research Council (CONICET),

C.C. 727, 1900 La Plata, Argentina.

Abstract

We give an exact information-theory treatment of the n-dimensional black-body radiation process in a non-extensive scenario. We develop a q generalization of the laws of i) Stefan Boltzmann, ii) Planck, and iii) Wien, and show that conventional, canonical results are obtained at temperatures above $1K$. Classical relationships between radiation, pressure, and internal energy are recovered (independently of the q-value). Analyzing the particles' density for $q \approx 1$ we see that the non-extensive parameter q introduces a fictitious chemical potential. We apply our results to experimental data on the cosmic microwave background and reproduce it with acceptable accuracy for different temperatures (each one associated to a particular q value).

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[*]E-mail: martinez@venus.fisica.unlp.edu.ar

[†]E-mail: pennini@venus.fisica.unlp.edu.ar

[‡]E-mail: plastino@venus.fisica.unlp.edu.ar

[§]E-mail: tessonec@venus.fisica.unlp.edu.ar

I. INTRODUCTION

Black-body radiation studies constitute a milestone in the history of Physics. Planck's law is satisfactorily accounted for by recourse to Bose-Einstein statistics and has been experimentally re-confirmed over and over for almost a century. However, in the last decade small deviations from this law have been detected in the cosmic microwave radiation [1]. In [2], its authors advance the hypothesis that these deviations could have arisen at the time of the matter-radiation de-coupling, and associate them to a non-extensive statistics environment. They argue in favor of such an hypothesis on the basis of the close relation that exists between long-range interactions and non-extensive scenarios [3,4]. The concomitant non-extensive thermostatistical treatment [3–12] is by now recognized as a new paradigm for statistical mechanical considerations. It revolves around the concept of Tsallis' information measure S_q [5], a generalization of the logarithmic Shannon's one. S_q is parameterized by a real index q and becomes identical to Shannon's measure when $q = 1$.

The study reported in [2] employs the so-called Curado-Tsallis un-normalized expectation values [10] in the limit $q \to 1$. Nowadays, it is believed that the Curado-Tsallis (CT) framework has been superseded by the so-called normalized or Tsallis-Mendes-Plastino (TMP) one [13,14], which seems to exhibit important advantages [7]. This normalized treatment, in turn, has been considerably improved by the so-called "Optimal Lagrange Multipliers" (OLM) approach [15]. It is, then, natural to revisit the problem from such a new viewpoint.

The OLM treatment can be recommended in view of the findings of [16–18] regarding the particular nature of the Lagrange multiplier associated to the temperature. Within an OLM context, non-extensivity is restricted just to the entropy. The internal energy remains extensive and the Lagrange multipliers conserve their traditional intensive character (allowing one to identify them with their thermodynamic counterpart). Moreover, the OLM approach unifies Tsallis and Rényi variational formalisms under a q-thermostatistics umbrella. The solutions to the concomitant OLM-Tsallis variational problem are also solutions for its Rényi-counterpart [19]. Since Rényi's entropy is extensive, one easily understands

thereby the OLM formalism's success in reproducing classical thermodynamic results in a simple and clean fashion [20,21].

We shall study here, from an OLM viewpoint, black-body radiation in equilibrium within an enclosure of volume V, with the goal of ascertaining the possible q-dependence of i) the Planck spectrum, ii) Stefan-Boltzmann's law, and iii) Wien's one. We will compare the ensuing results with those found in the literature [2,22–24] (all of them under Curado-Tsallis treatment) and with experimental data [1]. Due to the fact that most previous non-extensive treatments of quantal gases employ the so-called Factorization Approach (FA), and that this approximation underlies some of the preceding treatments of the black-body radiation problem, we will translate the FA into the language of a *normalized*-OLM Factorization Approach (OLM-FA). This in turn provides one with a q-normalized generalization of the particle's density expression for quantum gases.

The paper is organized as follows: In Sect. II we present a brief OLM primer. In Sect. III we obtain the OLM partition function. Sect. IV is devoted to the Stefan-Boltzmann law. We obtain the exact expression for the internal energy (IV A) and find that the Stefan-Bolzmann law holds for all temperatures except a small interval ($T \in [10^{-2}K, 1K]$). We also revisit pertinent antecedents in the literature of the non-extensive approach to the black-body problem (IV B). In Sect. V we perform an exact calculation of the energy density. Planck's law (V A) seems to be valid everywhere, with just slight variations in the shape of the associated curves. Following [2], the formalism is applied to the cosmic microwave radiation (V C) in order to search for deviations from Planck's law. Using the maximua of the energy density curves, we search for the q-generalization of Wien's law (V B) and find good agreement, save for a small temperature interval ($T \in [10^{-4}K, 10^{-2}K]$). Some conclusions are drawn in Sect. VI. In Appendix A we review the FA, introduce our new OLM-FA technique and apply it to the black-body radiation problem.

II. MAIN RESULTS OF THE OLM FORMALISM

For a most general quantal treatment, in a basis-independent way, consideration is required of the density operator $\hat{\rho}$ that maximizes Tsallis' entropy

$$\frac{S_q}{k} = \frac{1 - Tr(\hat{\rho}^q)}{q-1}, \tag{1}$$

subject to the M generalized expectation values $\left\langle \widehat{O}_j \right\rangle_q$, where $\widehat{O}_j$ $(j = 1, \ldots, M)$ denote the relevant observables.

Tsallis' normalized probability distribution [13] is obtained by following the well known MaxEnt route [25]. Instead of performing the variational treatment of Tsallis-Mendes-Plastino (TMP) [13], we will pursue the alternative path developed in [15]. One maximizes Tsallis' generalized entropy given by Eq. (1) [5,6,8] subject to the constraints [5,15]

$$Tr(\hat{\rho}) = 1 \tag{2}$$

$$Tr\left[\hat{\rho}^q \left(\widehat{O}_j - \left\langle \widehat{O}_j \right\rangle_q\right)\right] = 0, \tag{3}$$

whose generalized expectation values [13]

$$\left\langle \widehat{O}_j \right\rangle_q = \frac{Tr(\hat{\rho}^q \widehat{O}_j)}{Tr(\hat{\rho}^q)}, \tag{4}$$

are (assumedly) a priori known. In contrast with the TMP-instance [13], the resulting density operator

$$\hat{\rho} = \frac{\hat{f}_q^{1/(1-q)}}{\bar{Z}_q}, \tag{5}$$

is not self referential [15]. In Eq. (5), $\{\lambda_j\}$ $(j = 1, ..., M)$ stands for the so-called "optimal Lagrange multipliers' set". The quantity $\hat{f}_q$ is known as the configurational characteristic and takes the form

$$\hat{f}_q = 1 - (1-q) \sum_j^M \lambda_j \left(\widehat{O}_j - \left\langle \widehat{O}_j \right\rangle_q\right), \tag{6}$$

if its argument is positive, while otherwise $\hat{f}_q = 0$ (cut-off condition [13]). Of course, $\bar{Z}_q$ is the partition function

$$\bar{Z}_q = Tr \hat{f}_q^{1/(1-q)}. \tag{7}$$

It is shown in Ref. [15] that, as a consequence of the normalization condition, one has

$$\mathcal{R}_q \equiv Tr \hat{f}_q^{q/(1-q)} = \bar{Z}_q, \tag{8}$$

which allows one to write Tsallis's entropy as

$$S_q = k \ln_q \bar{Z}_q, \tag{9}$$

with $\ln_q x = (1 - x^{1-q})/(q-1)$. These results coincide with those of TMP [13]. Using Eq. (8), the connection between the OLM's set and the TMP's Lagrange multipliers set can be written as [15]

$$\lambda_j^{TMP} = \bar{Z}_q^{1-q} \lambda_j. \tag{10}$$

The TMP's Lagrange multipliers are not intensive quantities [16,18]. They do not, as a consequence, have a simple physical interpretation. To the contrary, the OLM multipliers *are* intensive [16,18], a fact that can be easily explained noticing that they are the natural Lagrange multipliers of a Rényi's variational approach [19].

If we define now

$$\ln Z_q = \ln \bar{Z}_q - \sum_j \lambda_j \ \left\langle \hat{O}_j \right\rangle_q, \tag{11}$$

we are straightforwardly led to [16]

$$\frac{\partial}{\partial \left\langle \hat{O}_j \right\rangle_q} \left(\ln \bar{Z}_q \right) = \lambda_j, \tag{12}$$

$$\frac{\partial}{\partial \lambda_j} \left(\ln Z_q \right) = - \left\langle \hat{O}_j \right\rangle_q. \tag{13}$$

Equations (12) and (13) are fundamental information theory (IT) relations for formulating Jaynes' version of statistical mechanics [25]. Due to Eqs. (9) and (10), the IT relation (12) leads straightforwardly to the well known expression [13]

$$\frac{\partial}{\partial \left\langle \hat{O}_j \right\rangle_q} \left(\frac{S_q}{k} \right) = \lambda_j^{TMP}. \tag{14}$$

III. PARTITION FUNCTION AND RADIATION PRESSURE

We will introduce now the exact (OLM) black-body radiation treatment. The chemical potential is taken, of course, equal to zero (Grand Canonical ensemble with $\mu = 0$) and, in looking for the equilibrium properties we consider that the appropriate thermodynamical variables are, as customary, the volume V, and the temperature T [26]. Consider first the standard situation $q = 1$.

The Hamiltonian of the electromagnetic field, in which there are $n_{\mathbf{k},\epsilon}$ photons of momentum $\mathbf{k}$ and polarization ϵ, is given by

$$\widehat{\mathcal{H}} = \sum_{k,\epsilon} \hbar\omega \hat{n}_{k,\epsilon}, \tag{15}$$

where the frequency is $\omega = c\,|\mathbf{k}|$ and $n_{k,\epsilon} = 0, 1, 2, \ldots$, with no restrictions on $\{n_{\mathbf{k},\epsilon}\}$.

For a macroscopic volume V the density of states g in an n-dimensional space is $g_n(\omega) = A_n\omega^{n-1}$, with

$$A_n = \frac{2\tau_n V}{(4\pi c^2)^{n/2}\Gamma(n/2)}, \tag{16}$$

where $\tau_n = n - 1$ is the number of linearly-independent polarizations.

The partition function

$$Z_1 = Tr\left(e^{-\beta\widehat{\mathcal{H}}}\right) \tag{17}$$

can be written as

$$Z_1 = \exp\left\{\int_0^\infty d\omega g_n(\omega) \ln\left[1 - \exp\left(-\beta\hbar\omega\right)\right]\right\} = e^{\xi_n}, \tag{18}$$

where

$$\xi_n = \frac{I_n A_n}{(\hbar\beta)^n}, \tag{19}$$

and

$$I_n = -\int_0^\infty dx x^{n-1} \ln\left(1 - e^{-x}\right) = \Gamma(n)\zeta(n+1), \tag{20}$$

with ζ standing for the Riemann function and Γ for the Gamma function.

The OLM-Tsallis generalized configurational characteristic will be (Cf. Eq. (6))

$$\hat{f}_q = 1 - (1-q)\beta(\widehat{H} - U_q), \tag{21}$$

where $\widehat{H}$ is the Hamiltonian given by Eq. (15), U_q is the mean energy introduced in Eq. (4), and $\beta = 1/kT$.

With the aim of calculating $\bar{Z}_q$ as defined by Eq. (7), we follow the steps of Ref. [22] and use the integral (Gamma) definition given by the relation [29]

$$b^{z-1} = \begin{cases} \frac{\Gamma(z)}{2\pi} \int\limits_{-\infty}^{\infty} dt \frac{e^{(1+it)b}}{(1+it)^z} & \text{for } b > 0 \\ 0 & \text{for } b \leq 0, \end{cases} \tag{22}$$

with $\mathrm{Re}(z) > 0$, and $-\pi/2 < \arg(a+it) < \pi/2$.

If we set $b = f_q$ (the cut-off condition is naturally fulfilled [7]) and $z = 1/(1-q)+1$ ($\mathrm{Re}(z) > 0$, so that either $q > 2$ or $q < 1$), the generalized partition function adopts the appearance

$$\bar{Z}_q(U_q) = \int\limits_{-\infty}^{\infty} dt K_q(t) Z_1(\tilde{\beta}), \tag{23}$$

with Z_1 given by Eq. (18),

$$K_q(t) = \frac{\Gamma\left[(2-q)/(1-q)\right] \exp(1+it) e^{\tilde{\beta} U_q}}{2\pi(1+it)^{(2-q)/(1-q)}}, \tag{24}$$

and

$$\tilde{\beta} = (1+it)(1-q)\beta. \tag{25}$$

In order to evaluate the integral in Eq. (23), we expand the exponential and obtain

$$\bar{Z}_q(U_q) = \frac{\Gamma[(2-q)/(1-q)]}{2\pi} \sum_{m=0}^{\infty} \frac{\xi_n^m}{m!} (1-q)^{-nm} \int\limits_{-\infty}^{\infty} dt \frac{e^{(1+(1-q)\beta U_q)(1+it)}}{(1+it)^{\frac{2-q}{1-q}+nm}}, \tag{26}$$

where ξ_n is given by Eq. (19).

Using again Eq. (22), with $b = 1 + (1-q)\beta U_q$ and $z = (2-q)/(1-q) + nm$, we arrive at

$$\bar{Z}_q(U_q) = \Gamma\left[(2-q)/(1-q)\right]\left[1+(1-q)\beta U_q\right]^{1/(1-q)} \sum_{m=0}^{\infty} B_m \Gamma^{-1}\left[(2-q)/(1-q)+nm\right], \quad (27)$$

where

$$B_m = \frac{\xi_n^m}{m!}\frac{\left[1+(1-q)\beta U_q\right]^{nm}}{(1-q)^{mn}}. \quad (28)$$

Notice that an additional, cut-off-like condition must be considered, namely,

$$1+(1-q)\beta U_q > 0, \;\; otherwise, \;\; \bar{Z}_q(U_q) = 0.$$

A similar path can be followed in order to obtain the quantity $\mathcal{R}_q$ (introduced in Eq. (8)), which will read

$$\mathcal{R}_q(U_q) = \Gamma\left[1/(1-q)\right]\left[1+(1-q)\beta U_q\right]^{q/(1-q)} \sum_{m=0}^{\infty} B_m \Gamma^{-1}\left[1/(1-q)+nm\right], \quad (29)$$

although in this case the allowed interval of q-values is reduced to $0 < q < 1$. Note that this permissible interval of q-values respects the new cut-off condition introduced above.

Once we have $\bar{Z}_q$, the radiation pressure is easily obtained by applying Eq. (12) to $(\beta P, V)$ [26], i.e.,

$$\beta P = \frac{\partial}{\partial V}\ln \bar{Z}_q = \frac{1}{V}\frac{\sum_m m B_m \Gamma^{-1}\left((2-q)/(1-q)+nm\right)}{\sum_m B_m \Gamma^{-1}\left((2-q)/(1-q)+nm\right)}. \quad (30)$$

Due to the fact that we have set $\mu = 0$, the quantity $\bar{Z}_q$ of Eq. (27) does not depend on the mean number of particles N_q. One has

$$\mu = \beta^{-1}\frac{\partial}{\partial N_q}\ln \bar{Z}_q = 0. \quad (31)$$

IV. STEFAN-BOLZMANN'S LAW

A. Exact OLM treatment

As in the previous section, the generalized internal energy of the black-body radiation is obtained from Eq. (4) by specializing the problem to the Grand Canonical ensemble with $\mu = 0$

$$U_q = \mathcal{R}_q^{-1} Tr\left(\hat{f}_q^{q/(1-q)} \widehat{H}\right), \tag{32}$$

with $\hat{f}_q$ given by Eq. (21).

Consider now the trace's content. Using (22) we can find

$$U_q = \mathcal{R}_q^{-1} \frac{\Gamma[1/(1-q)]}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1+(1-q)\beta U_q]}}{(1+it)^{\frac{1}{1-q}}} Tr\left(Z_1(\tilde{\beta})\widehat{H}\right), \tag{33}$$

where $\tilde{\beta}$ is given by Eq. (25). The permissible q-interval is $0 < q < 1$, due to restrictions posed by the Gamma integral representation. Taking advantage of the fact that

$$Tr\left[Z_1(\tilde{\beta})\widehat{H}\right] = -\frac{\partial Z_1(\tilde{\beta})}{\partial \tilde{\beta}} = \frac{n}{\tilde{\beta}} \sum_m \frac{\xi_n^{m+1}(\tilde{\beta})}{m!}, \tag{34}$$

we obtain

$$U_q = \mathcal{R}_q^{-1} \frac{n}{2\pi} \Gamma\left(\frac{1}{1-q}\right) \frac{\xi_n}{(1-q)^{n+1}\beta} \sum_{m=0}^{\infty} \frac{\xi_n^m}{m!} \frac{1}{(1-q)^{nm}} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1+(1-q)\beta U_q]}}{(1+it)^{\frac{1}{1-q}+n(1+m)+1}}, \tag{35}$$

and, using again Eq. (22), we obtain the mean energy expression

$$U_q = \frac{n\xi_n}{(1-q)^{n+1}\beta} \left[1+(1-q)\beta U_q\right]^{n+1} \frac{\sum\limits_{m=0}^{\infty} B_m \Gamma^{-1}[1/(1-q)+n(m+1)+1]}{\sum\limits_{m=0}^{\infty} B_m \Gamma^{-1}[1/(1-q)+nm]}. \tag{36}$$

Notice that the Tsallis'cut-off condition $1+(1-q)\beta U_q > 0$ is always satisfied. The series in Eq. (36) rapidly converges on account of the exponential-like factors B_m (Cf. Eq. (28)), weighted by inverse Gamma functions.

Lenzi *et al.* solved the q-black-body radiation problem *in exact* fashion in its Curado-Tsallis un-normalized version [22]. Since their resulting internal energy is not self referential (as it is in the TMP normalized instance) they were able to describe the asymptotic behavior for β in quite simple terms. *Here* we need to perform a more detailed analysis by considering different possibilities for the form that the product βU_q may take as $\beta \to \infty$. For instance, if we assume that i) βU_q goes over to a constant or ii) it is not bounded, the limiting process leads to incoherencies, while if we assume that, in Eq. (36), $\beta U_q \to 0$ when $\beta \to \infty$, $U_q \propto T^4$, as one has the right to expect.

As a consequence of the normalization condition given by Eq. (3), we know that $\mathcal{R}_q = \bar{Z}_q$ (see Eq. (8)). This relation allows us to look for alternative expressions for the relevant mean value. If we evaluate U_q in terms of $\bar{Z}_q$ we find

$$U_q = \frac{n}{\beta} \frac{\sum\limits_{m=0}^{\infty} m B_m \Gamma^{-1}\left((2-q)/(1-q)+nm\right)}{\sum\limits_{m=0}^{\infty} B_m \Gamma^{-1}\left((2-q)/(1-q)+nm\right)}. \tag{37}$$

By inspection of Eq. (30) one realizes that the traditional relation between internal energy and pressure still holds here

$$P = \frac{1}{n}\frac{U_q}{V}. \tag{38}$$

Eqs. (36) or (37) are recursive expressions that give the OLM version of Stefan-Boltzmann's law, which, for the $q=1$ case, reads

$$U = n\xi_n/\beta \propto T^{n+1}. \tag{39}$$

The present equations are to be tackled numerically. Fig. 1 depicts U_q as a function of T for different values of q and $n=3$ in a log-log scale, where U_q has been evaluated from Eq. (36). It is seen that Stefan-Boltzmann's law is reproduced by our formalism for a wide range of T-values. Violations are detected just for some special T-ranges (that depend on q), within the interval $10^{-2}K < T < 1K$.

With the present formalism the Stefan-Bolzmann constant becomes a function of i) q (namely, "σ_q") and ii) the relevant range of temperatures. Indeed, for temperatures below $10^{-2}K$ results will be markedly different from those obtained if we consider $T > 1K$ (see Fig. 2). It may be appreciated that, for these two Temperature ranges, the values of σ_q increase monotonically with q.

Looking for insights into the meaning of the "violation range" $10^{-2}K < T < 1K$ we considered the specific heat

$$C_q = \frac{dU_q}{dT}. \tag{40}$$

The concomitant results are plotted in Fig. 3. One sees that the violation range coincides with that of a constant specific heat, i.e., $U_q \propto T$. The specific heat curves' behavior is quite different from that associated with Gibbs' predictions, even for q values close to unity. By recourse to detailed numerical analysis one notes that the peak one observes is not a discontinuity but part of a smooth curve that reflects the typical behavior of a first excited energy level when that level lies too close to the ground state. The typical step-like form tends to disappear in the $q \to 1$ limit, just as if $1-q$ were a new "degree of freedom" of the system.

For more details on the transition between Gibbs' and Tsallis's statistics let us analyze the $q \to 1$ limit in Eq. (36). Using the fact that the pertinent series are highly convergent, we can try to recover analytically the above numerical results by keeping only the first order term of the pertinent series,

$$U_q \approx \frac{n\chi_q\xi_n}{\beta}\left[1+(1-q)\beta U_q\right]^{n+1}, \tag{41}$$

where $\chi_q = \Gamma[1/(1-q)](1-q)^{-(n+1)}\Gamma^{-1}[1/(1-q)+n+1]$. For $q \to 1$, $\chi_q \to 1$ and we see that Stefan-Boltzmann's law is recovered.

Now, to first order in $1-q$,

$$\left[1+(1-q)\beta U_q\right]^{n+1} \approx 1+(n+1)(1-q)\beta U_q,$$

so that, rearranging terms, Eq. (41) can be cast as

$$U_q \approx \frac{n\chi_q\xi_n\beta^{-1}}{1-(1-q)(n+1)n\chi_q\xi_n}. \tag{42}$$

The change of behavior we are interested in is better observed with reference to Eq. (37), by keeping only first order terms in the series expansion. We find

$$U_q \approx \frac{n}{\beta}\frac{B_1\Gamma^{-1}[(2-q)/(1-q)+n]}{\Gamma^{-1}[(2-q)/(1-q)]+B_1\Gamma^{-1}[(2-q)/(1-q)+n]}, \tag{43}$$

where

$$B_1 = \xi_n\frac{\left[1+(1-q)\beta U_q\right]^n}{(1-q)^n}. \tag{44}$$

It is clear that, for βU_q such that the first term in the denominator of Eq. (43) dominates, $U_q \propto T^{n+1}$. When the second term is dominant, instead, then $U_q \propto T$, in agreement with our numerical results. Eq. (42) displays a similar behavior, although its validity is restricted to the $q \to 1$ limit. This prevents the second term in the denominator from being dominant, a fact reflected in Fig. 1, where the linear dependence is seen to fade away.

B. Comparison with previous non-extensive results

The Stefan-Boltzmann's law was first discussed within the Tsallis' non-extensive framework in [2]. In this paper the authors employ the so-called Curado-Tsallis un-normalized expectation values [5] and work in the limit $q \to 1$: a first order approximation in $1 - q$ is used for the partition function in order to study the cosmic microwaves' background (in Sect. V C we will apply the results of the present work to the same data set). In a subsequent effort, Lenzi *et al.* [22] advanced an exact treatment for the same problem, also within the Curado-Tsallis framework. A third relevant work is that of Tirnakli *et al.* [24], that compared the exact nonextensive treatment of the problem with the FA one, including the un-normalized asymptotic approach (AA) already introduced in [2].

The different ensuing expressions for the pertinent à la Stefan-Bolzmann laws are given in Table A 2 for $n = 3$. The traditional Gibbs expression can be found in the first row, while the second and third are, respectively, the Curado-Tsallis' and OLM (exact) results. The TMP solution can be read off the fourth row, where we have simply used Eq. (10) for β. The last three rows are devoted to the $q \to 1$ limit. The first one contains the factorization approach result, the second is the our new, OLM factorization approach expression, and, finally, the last one gives the exact OLM treatment in this limit.

Inspection of these expressions allows one to appreciate that the OLM-FA seems to inherit characteristics of both the FA and the OLM treatments for $q \approx 1$. The OLM-FA numerator coincides with that of the FA approach, while the denominator resembles the OLM one (except for the sign). Indeed, the concomitant integrals are identical here to those

of the un-normalized treatment.

V. PLANCK AND WIEN LAWS

A. Planck's law: exact OLM treatment

The generalized spectral energy distribution of black-body radiation u_q is defined by the integral

$$U_q = \int_0^\infty d\omega u_q. \tag{45}$$

In order to obtain u_q we need to analyze, again, the trace's argument in the expression for U_q. We will follow the path already pursued above, but without integrating the $q = 1$-like term over frequencies. We obtain

$$Tr\left(Z_1(\tilde{\beta})\widehat{H}\right) = \hbar A_n e^{\xi_n} \int_0^\infty d\omega \omega^n \frac{e^{-\tilde{\beta}\hbar\omega}}{1 - e^{-\tilde{\beta}\hbar\omega}}, \tag{46}$$

that, by recourse to the identity

$$\frac{1}{1 - e^{-\tilde{\beta}\hbar\omega}} = \sum_{s=0}^\infty \left(e^{-\tilde{\beta}\hbar\omega}\right)^s, \tag{47}$$

allows us to cast U_q in the fashion

$$U_q = \mathcal{R}_q^{-1} \frac{\Gamma[1/(1-q)]}{2\pi} \hbar A_n \int_0^\infty d\omega \omega^n \sum_{s=0}^\infty \sum_{m=0}^\infty \frac{\xi_n^m}{m!} \int_{-\infty}^\infty dt \frac{e^{(1+it)[1-(1-q)\beta[\hbar\omega(1+s)-U_q]]}}{(1+it)^{\frac{1}{1-q}+nm}}. \tag{48}$$

With Eq. (22) the integral above is easily evaluated, and we obtain

$$U_q = \mathcal{R}_q^{-1} \Gamma[1/(1-q)] \hbar A_n \int_0^\infty d\omega \omega^n \sum_{s=0}^\infty \sum_{m=0}^\infty \frac{\xi_n^m}{m!} \frac{[1-(1-q)\beta[\hbar\omega(1+s) - U_q]]^{\frac{q}{1-q}+nm}}{(1-q)^{mn}\Gamma[1/(1-q)+nm]}. \tag{49}$$

According to Eq. (29), the energy density defined by Eq. (45) is

$$u_q(\omega) = \hbar A_n \omega^n \frac{\sum\limits_{m=0}^\infty B_m S_m \Gamma^{-1}[1/(1-q)+nm]}{\sum\limits_{m=0}^\infty B_m \Gamma^{-1}[1/(1-q)+nm]}, \tag{50}$$

where B_m is given by Eq. (28) while S_m reads

$$S_m = \sum_{s=1}^{s_q} [1 - (1-q)\beta_q \hbar\omega s]^{\frac{q}{1-q}+nm}\,, \tag{51}$$

where

$$s_q = \left[\frac{1}{(1-q)\beta_q \hbar\omega}\right], \tag{52}$$

and

$$\beta_q = \frac{\beta}{1 + (1-q)\beta U_q}. \tag{53}$$

Notice that a new cut-off condition has been introduced, namely, $1 - (1-q)\beta_q\hbar\omega s > 0$, that transforms the original series into a finite sum.

Since one knows U_q from Eq. (36), an expression for u_q is easily obtained. Notice that it is not self referential. The shape of the resulting curves is similar to those arising from the traditional treatment, even for T values where one detects q-violations to Plank's law. Fig. 4 depicts u_q *vs* ω for different q values. The maximum's values do not coincide with the ones of the canonical Gibb's approach when q differs from unity. In Fig. 4 (a) it becomes apparent that results for $q = 1$ and for $q = 0.98$ greatly differ. The difference seems to become less important between results for $q = 0.9$ and those for $q = 0$, as can be seen in (b). Another point to be stressed is that the maximum's position will also depend on q, not just on T, as in the traditional Wien's law.

We have seen from Fig. 4 that Eq. (50) yields results that quite resemble the ones given by Planck's law. It is then reasonable to look for a perturbative expansion in $1 - q$. The series over the m factor in Eq. (50) rapidly converges. Only the first terms are important. In the limit $q \to 1$, the exponent $q/(1-q) + nm \approx q/(1-q)$ in Eq. (51), and the finite sum becomes a series expansion. We have

$$u_q \approx \hbar A_n \omega^n S, \tag{54}$$

where

$$S = \sum_{s=1}^{\infty} [1 - (1-q)\beta_q\hbar\omega s]^{\frac{q}{1-q}}\bigg|_{q\to 1} \approx \sum_{s=0}^{\infty} \exp\left[-q\beta_q\hbar\omega(1+s)\right], \tag{55}$$

a power series in s of guaranteed convergence, i.e.,

$$S \approx \frac{1}{e^{q\beta_q\hbar\omega} - 1}, \tag{56}$$

so that replacement into Eq. (54) yields

$$u_q \approx \frac{\hbar A_n \omega^n}{e^{q\beta_q\hbar\omega} - 1}, \tag{57}$$

a first order correction to the Planck law. The classical result is attained for $q \to 1$. Eq. (57) provides one, then, with an approximate energy density.

Following the standard text-book treatment [26], we can define the particle-density as

$$n_q = (\hbar A_n \omega^n)^{-1} u_q, \tag{58}$$

which, from Eq. (57) can be written, to first order in $1-q$ as

$$n_q = \frac{1}{e^{q\beta_q\epsilon} - 1}, \tag{59}$$

with $\epsilon = \hbar\omega$.

We have encountered an alternative expression for the boson particle-density of the black-body radiation, to be compared to the ones that result from either the FA or OLM-FA (see Apendix A). The correct value for the $q \to 1$ limit is obtained.

Consider now the specific heat curves of Sect. IV A. From Eq. (53) we see that, to first order in $1-q$ we have

$$\beta_q \approx \beta[1 - (1-q)\beta U_q], \tag{60}$$

that, replaced into Eq. (59) and using the fact that $q = 1 - (1-q)$ leads to (keeping only terms of order $1-q$)

$$n_q \approx \frac{1}{e^{\beta(\epsilon - \mu_q^*)} - 1}, \tag{61}$$

where

$$\mu_q^* = (1-q)\epsilon(1 + \beta U_q) \tag{62}$$

plays the role of a fictitious q-chemical potential, such that $\mu_q^* \to 0$ for $q \to 1$. Of course, the "true" *physical* chemical potential vanishes identically (see Sect. III). A sort of fictitious q-Bose-Einstein condensation effect might seem to be implied by the presence of this pseudo-chemical potential, a point that deserves further careful exploration, to be addressed in a future work.

B. Wien law

A nice result of the traditional thermostatistics is the linear relation that exists between the frequency ω^* that maximizes the energy density spectrum, on the one hand, and the temperature on the other one

$$\omega^* \propto T. \tag{63}$$

This relation is known as Wien's law.

According to the results depicted in Fig. 4 (see Sect. V A), deviations of the these maximum values from what we expect from them according to the above relation will depend on q. If we perform the pertinent derivatives in Eq. (50) we obtain

$$\sum_{m=0}^{\infty} B_m \Gamma^{-1} \left[1/(1-q) + nm\right] \left(\frac{n}{\omega} S_m + \frac{dS_m}{d\omega} \right) \Bigg|_{\omega^*} = 0, \tag{64}$$

where S_m depends in a non trivial manner on ω due to the cut-off condition (see Eq. (51): the cut-off condition "cuts" the series at an s_q that depends on ω).

Numerical calculations of Eq. (64) are depicted in Fig. 5. It can be seen that the linear dependence is respected over temperatures above $10^{-2}K$ and below $10^{-4}K$. There is a "violation range" the T-zone ($10^{-4} < T < 10^{-2}$). This implies lower values of T than those that violate Stefan Boltzmann's law. Departures from Wien's law are observed even for $q-$values as close to unity as $q = 0.98$. Using now Eq. (57) as a starting point we also get first order corrections in $1-q$ to the Wien law. We can locate the maximum with respect to ω of Eq. (57) with the auxiliary definition

$$x = \frac{\omega q \beta \hbar}{1-(1-q)\beta U_q}, \tag{65}$$

so that we immediately find

$$e^{-x} + \frac{x}{n} = 1, \tag{66}$$

whose solution is a constant, b, for each fixed value of n. For $n = 3$ we find, for instance, $b = 2.82$. The maximum of $u_q(\omega)$ for different T's is located at distinct frequencies according to

$$\omega_i = \frac{bk}{q\hbar} T_i - \frac{1-q}{q} \frac{U_q}{\hbar}, \tag{67}$$

a first order correction to Wien's law.

C. Microwave Cosmic Background Radiation

The OLM formalism can be applied to the analysis of the cosmic microwave radiation in order to search for putative deviations from Planck's law. The best source of data on this topic comes from the COBE satellite (Far-infrared Absolute Spectrophotometer (FIRAS)). The ensuing brightness, when compared to a Planck spectrum at the temperature $T = 2.72584 \pm 0.0005K$ shows significant deviations from the expected form (the χ^2 of the fit to a Planck spectrum is rather large).

FIRAS measures the *differential* spectrum between the cosmic background and an internal reference adjusted to approximately $2.7K$ so as to i) avoid problems with imperfect emissivity and ii) limit the dynamic range of the instrument. In addition, an external blackbody is used to calibrate (according to Gibbs' statistics) the gain of the instrument with a temperature in the range of $2K$ up to $25K$.

This is the kind of data that will be of interest in order to verify the predictions of the OLM formalism. If we assume the cosmic background does not obey the usual Planck's law, but the OLM one, we can check the concomitant differences and contrast the ensuing values with those obtained by FIRAS. The results are depicted in Fig. 6. We plot the

best fits for four different values of q. The range of temperatures that allow for the best fitting is also given. In all the graphs it is apparent that deviations are well predicted if we assume a statistics with $q \neq 1$ for the cosmic background. As q deviates from unity higher temperatures are predicted, in agreement with the results displayed in Fig. 4. We see there that the spectrum maximum's value becomes smaller for a given, fixed temperature when q deviates from unity.

The experimental data depicted in Fig. 6 seems to imply that the statistics ruling the cosmic microwave radiation is different from that of the conventional black-body instance. This constitutes evidence for the existence of alternative statistics, and tells us that the statistics behind these phenomena seems to possess an additional, q-degree of freedom.

VI. CONCLUSIONS

Within the q-Thermostatistics framework we perform an exact analysis of the n-dimensional black-body radiation process. We employ to such effect both Tsallis' and Rényi's entropies, within the range $0 < q < 1$. The new theoretical ingredient here is the so-called OLM approach to non-extensive thermostatistics [15].

We develop a q generalization of several laws: Stefan Boltzmann's , Planck's, and Wien's. We find the the conventional behavior still obtains for temperatures above $1K$, although there is a q dependence in the appropriate proportionally constants. We recover the traditional relationship between radiation pressure and internal energy using the OLM formalism.

We apply the formalism to experimental data on the cosmic microwaves' background and reproduce it with acceptable accuracy even for q-values that appreciably differ from unity. The larger $|1 - q|$ is, the higher the predicted equilibrium cosmic temperature.

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REFERENCES

[1] J. C. Mather et *al.*, *Astrophys. J.* **420** (1994) 439.

[2] C. Tsallis, F. C. Sà Barreto, and E. D. Loh *Physical Review E* **52** (1995) 1447.

[3] A. R. Plastino and A. Plastino, *Braz. J. of Phys.* **29** (1999) 79.

[4] A. R. Plastino, A. Plastino, *Physics Letters A* **174** (1993) 834.

[5] C. Tsallis, *Braz. J. of Phys.* **29** (1999) 1, and references therein. See also http://www.sbf.if.usp.br pages/Journals/BJP/Vol129/Num1/index.htm

[6] C. Tsallis, *Chaos, Solitons, and Fractals* **6** (1995) 539, and references therein; an updated bibliography can be found in http://tsallis.cat.cbpf.br/biblio.htm

[7] C. Tsallis, Nonextensive statistical mechanics and thermodynamics: Historical background and present status, in "Nonextensive Statistical Mechanics and its Applications", eds. S. Abe and Y. Okamoto, "Lecture Notes in Physics" (Springer-Verlag, Berlin, 2001).

[8] C. Tsallis, *J. Stat. Phys.* **52** (1988) 479.

[9] C. Tsallis, *Physics World 10* (July 1997) 42.

[10] E. M. F. Curado and C. Tsallis, *J. Phys. A* **24** (1991) L69; Corrigenda: **24** (1991) 3187 and **25** (1992) 1019.

[11] A. Plastino and A. R. Plastino, *Braz. J. of Phys.* **29** (1999) 50.

[12] A. R. Plastino and A. Plastino, *Phys. Lett. A* **177** (1993) 177.

[13] C. Tsallis, R. S. Mendes, and A. R. Plastino, *Physica A* **261** (1998) 534.

[14] F. Pennini, A. R. Plastino and A. Plastino, *Physica A* **258** (1998) 446.

[15] S. Martínez, F. Nicolás, F. Pennini and A. Plastino, *Physica A* **286** (2000) 489.

[16] S. Martínez, F. Pennini and A. Plastino, *Physics Letters A* **282** (2001) 263 .

[17] S. K. Rama, *Phys. Lett. A* **276** (2000) 1.

[18] S. Martínez, F. Pennini, and A. Plastino. *Physica A : Proceedings of the IUPAP International Conference on New Trends in the Fractal Aspects of Complex Systems*, **295** (2001) 246.

[19] E. K. Lenzi, R. S. Mendes, L. R. da Silva, *Physica A* **280** (2000) 337.

[20] S. Martínez, F. Pennini, and A. Plastino. *Physics Letters A*, **278** (2000) 47.

[21] Sumiyoshi Abe, S. Martínez, F. Pennini, and A. Plastino. *Physics Letters A*, **278** (2001) 249.

[22] E. K. Lenzi, R. S. Mendes, *Physics Letters A* **250** (1998) 270.

[23] F. Büyükkilić and D. Demirham, *Phys. Lett. A* **181** (1993) 24; F. Büyükkilić, D. Demirham and A. Gulec, *Phys. Lett. A* **197** (1995) 209.

[24] U. Tirnakli, D. F. Torres, *Eur. Phys. J. B* **14** (2000) 691.

[25] E. T. Jaynes in *Statistical Physics*, ed. W. K. Ford (Benjamin, NY, 1963); A. Katz, *Statistical Mechanics*, (Freeman, San Francisco, 1967).

[26] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987) pp. 278-283.

[27] R. K. Pathria, *Statistical Mechanics*, (Pergamon, New York, 1985).

[28] F. Reif, Statistical thermal physics, (MacGraw-Hill, E.U. Condon editor, University of Colorado 1965), p. 388.

[29] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals Series and Products (Academic Press, New York, 1980) p. 935.

APPENDIX A: THE OLM FACTORIZATION APPROXIMATION

1. Un-normalized Factorization Approximation (FA)

In order to facilitate the reader's task we review now the factorization approach (FA) treatment, due to Büyükkilic *et al.*, who tackled in Ref. [23] the quantum ideal gas problem in a grand canonical scenario using the CT-Tsallis formalism. Tsallis' entropy is thereby maximized subject to the un-normalized constraints

$$Tr\rho^q \hat{H} = U_q \tag{A1}$$

$$Tr\rho^q \hat{N} = N_q. \tag{A2}$$

The pertinent partition function is

$$Z_q^{CT} = \sum_i [1-(1-q)\sum_j n_{ij} x'_j]^{1/(1-q)}, \tag{A3}$$

where n_{ij} are the occupation numbers of the level j (with single particle energy ϵ_j) for a given i-microscopic configuration and

$$x'_j = \beta^{CT}(\epsilon_j - \mu^{CT}), \tag{A4}$$

with β^{CT} the inverse temperature and μ^{CT} the chemical potential.

According to [23], "there is no restriction on the summation and thus one an factorise a product of factors, one for each one-particle state, since particle are regarded statistically independent" . Therefore (the essential point of the FA approach), the partition function Z_q^{CT} given by Eq. (23) can be factorized. Each factor corresponds to a single particle state

$$Z_q^{CT} \approx \prod_{j=0}^{\infty} \sum_{i=0}^{w} \left[1-(1-q) n_{ij} x'_j\right]^{1/(1-q)}. \tag{A5}$$

The average occupation number $\langle n_j \rangle_{FA}$ of the state j becomes then [23]

$$\langle n_j \rangle_{FA} \approx \frac{1}{[1-(1-q)x'_j]^{1/(1-q)} \mp 1}, \tag{A6}$$

which is a non-extensive, FA-Tsallis generalization of the Bose-Einstein (Fermi-Dirac) distribution.

2. The normalized OLM treatment

An alternative treatment to the factorization approach using the TMP formalism yields self-referential probabilities. The OLM procedure, instead, is not afflicted by such a problem. One again maximizes Tsallis' generalized entropy given by Eq. (1) subject to the diagonal normalized constraints for the Grand Canonical Ensemble, which results in a partition function of the form

$$\bar{Z}_q = e_q^{x_q} \sum_i \left[1 - (1-q) \sum_j n_{ij} x_j \right]^{1/(1-q)} , \tag{A7}$$

where $e_q^x = [1 + (1-q)x]^{1/(1-q)}$ and, instead of x'_j (Eq. (A4)), we have

$$x_j = \frac{\beta(\epsilon_j - \mu)}{1 + (1-q)x_q}, \tag{A8}$$

with

$$x_q = \beta(U_q - \mu N_q). \tag{A9}$$

Note that $x_j \to x'_j$ for $q \to 1$. The conventional limit is the same for both treatments, if one remembers that the un-normalized methodology uses a partition function Z_q^{CT}, while the normalized treatment deals with a *bar* partition function $\bar{Z}_q$ (see Eq. (11)). The concomitant $q = 1$-limits are related according to $Z_1 = e^{-\beta(U - \mu N)} \bar{Z}_1$.

Proceeding once again as in Ref. [23], i.e., neglecting correlations between particles and regarding states of different particles as statistically independent, the partition function $\bar{Z}_q$ can be factorized. Each factor corresponds to a single particle state,

$$\bar{Z}_q \approx e_q^{x_q} \prod_j \sum_i \left[1 - (1-q) n_{ij} x_j\right]^{1/(1-q)} . \tag{A10}$$

The present expression for $\bar{Z}_q$ is *formally* identical to that encountered in Ref. [23] (see Eq. (A5)), save for a multiplicative factor, which does not contribute to the ensuing microscopic probabilities. Thus, according to Ref. [23], the average value $\langle n_j \rangle_{NFA}$ is now

$$\langle n_j \rangle_{NFA} \approx \frac{1}{[1 - (1-q)x_j]^{1/(1-q)} \mp 1} \tag{A11}$$

with x_j given by Eq. (A9).

We introduce here the normalized factorization approach (NFA) black-body's treatment. The mean occupation number is given by Eq. (A11) with $\mu = 0$ (the chemical potential vanishes because we are working with photons). In the continuum limit $\epsilon(\omega) = \hbar\omega$ (ω is the frequency), and the particle-density will be given by

$$\langle n(\omega)\rangle_{NFA} \approx \frac{1}{[1-(1-q)x'(\omega)]^{1/(1-q)} - 1}, \tag{A12}$$

with

$$x'(\omega) = \frac{\beta\hbar\omega}{1+(1-q)\beta U_q}. \tag{A13}$$

In the three dimensional case the mean occupation number is connected with the energy-density according to

$$u_q(\omega) = \frac{\hbar V}{\pi^2 c^3}\omega^3 \left\langle n(\omega)\right\rangle_q. \tag{A14}$$

Keeping only first order terms in $1-q$ in Eq. (A14), integrating over frequencies, and re-arranging terms, the internal energy reads

$$U_q^{NFA} = \sigma T^4 \frac{1-(1-q)\theta}{1+\frac{4}{k}(1-q)\sigma T^3}, \tag{A15}$$

where $\sigma = (\pi^2 k^4 V)/(15\hbar^3 c^3)$ (with c, the light's speed) is in closed connection with the Stefan-Boltzmann constant $(c\sigma)/(4V)$ [28].

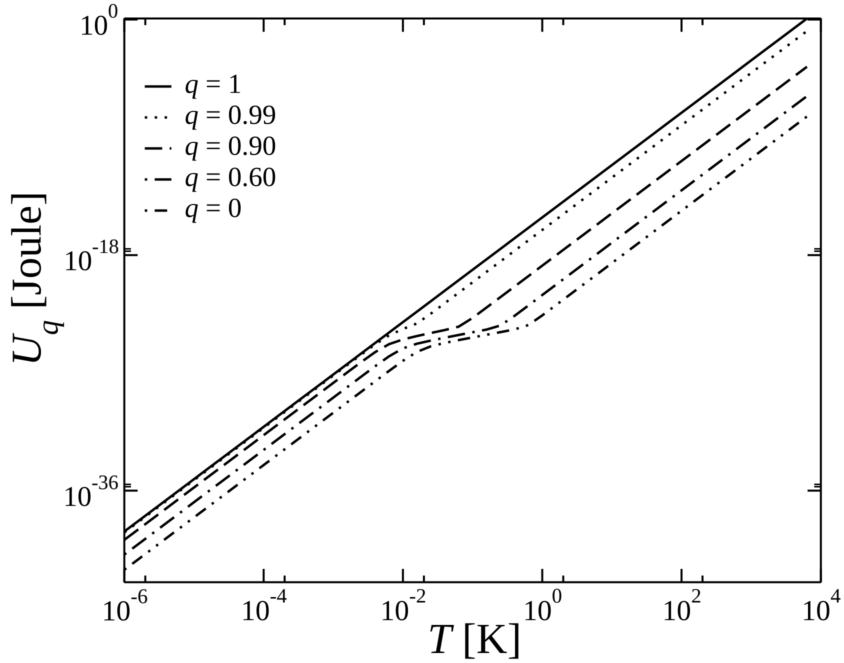

FIG. 1. Internal energy, U_q, of a three-dimensional system as a function of Temperature $T = 1/\beta$ for different values of the non-extensivity parameter q for a box whose volume is 1m^3.

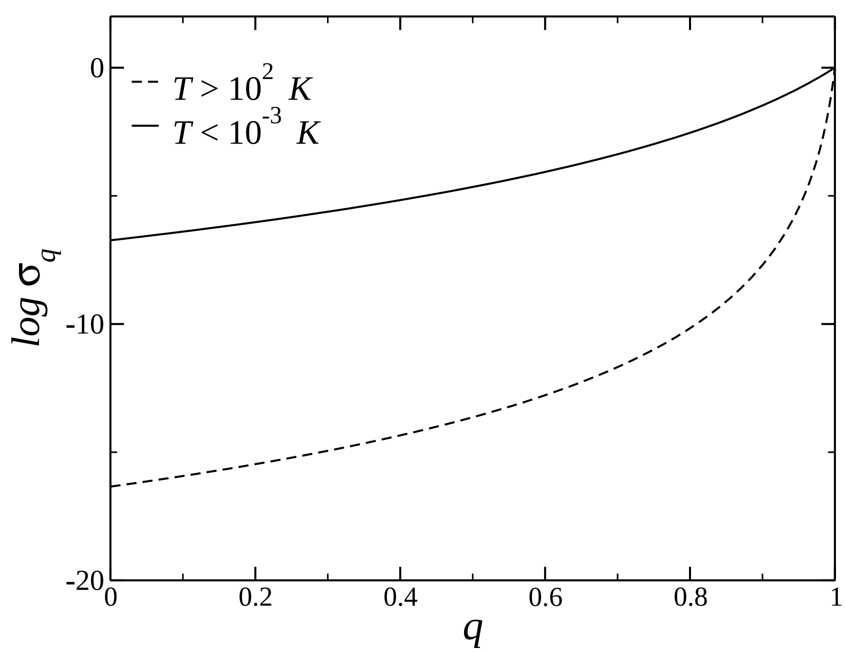

FIG. 2. Generalized Stefan constant σ_q as a function the index q. It is shown $\ln(\sigma_q/\sigma)$ vs q with $\sigma = 5.67051 \cdot 10^{-8}\text{Watt}/(K^4\text{Meter}^2)$ the Stefan constant for $q = 1$. See inline text for details.

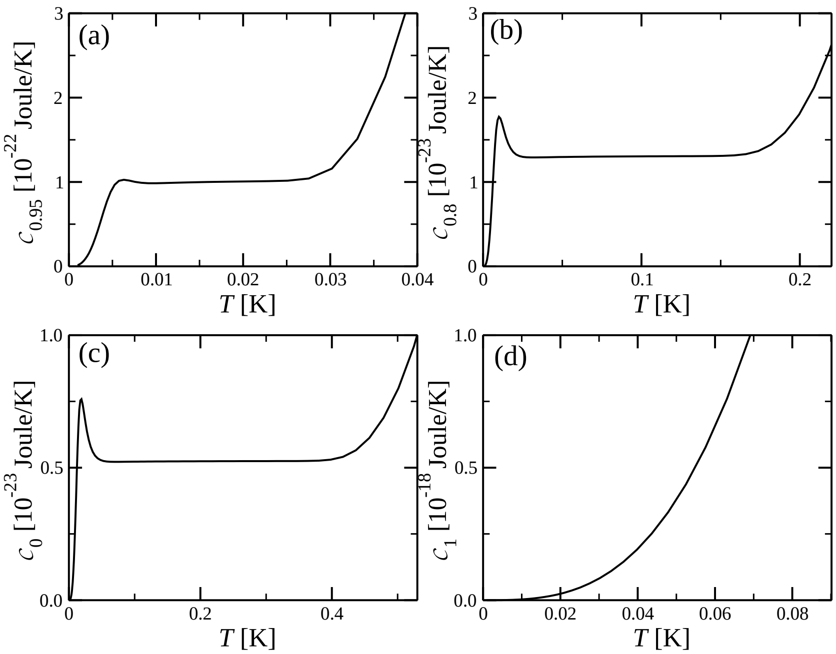

FIG. 3. Specific heat, $\mathcal{C}_q$, of a three-dimensional box of volume 1m^3, as a function of Temperature T, for different values of the non-extensivity parameter q. The plots show the arising results for (a) $q = 0.98$, (b) $q = 0.8$, (c) $q = 0$, (b) $q = 1$.

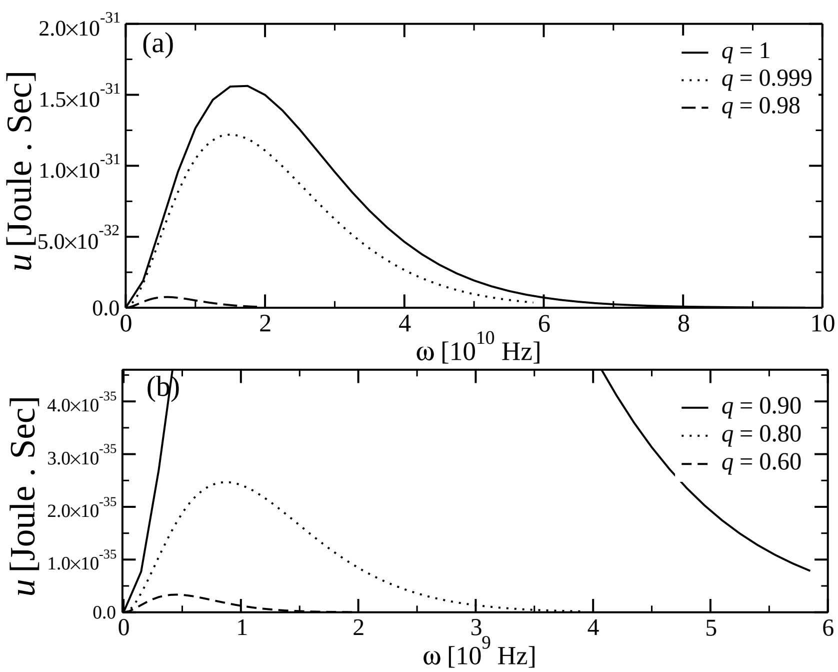

FIG. 4. Energy density u_q as a function of the frequency ω. All the curves were obtained for a temperature $T = 0.1K$, and $V = 1\text{m}^3$. Inset (a) shows the results for the limit $q \to 1$, whereas in the inset (b) the results for obtained otherwise are depicted.

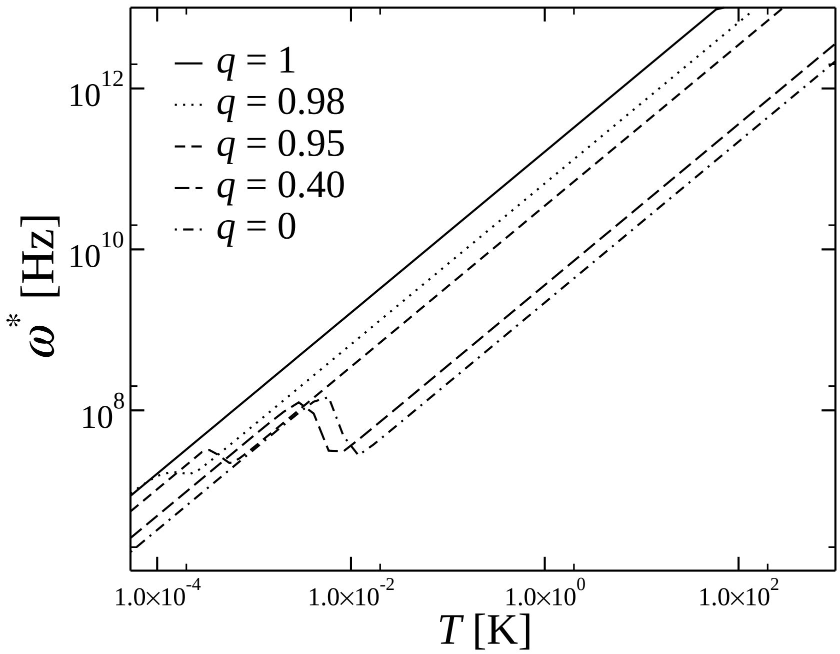

FIG. 5. Frequency ω^* for which the energy density reaches its maximum, as a function of temperature T, for different values of q. As in the previous figures, $n = 3$.

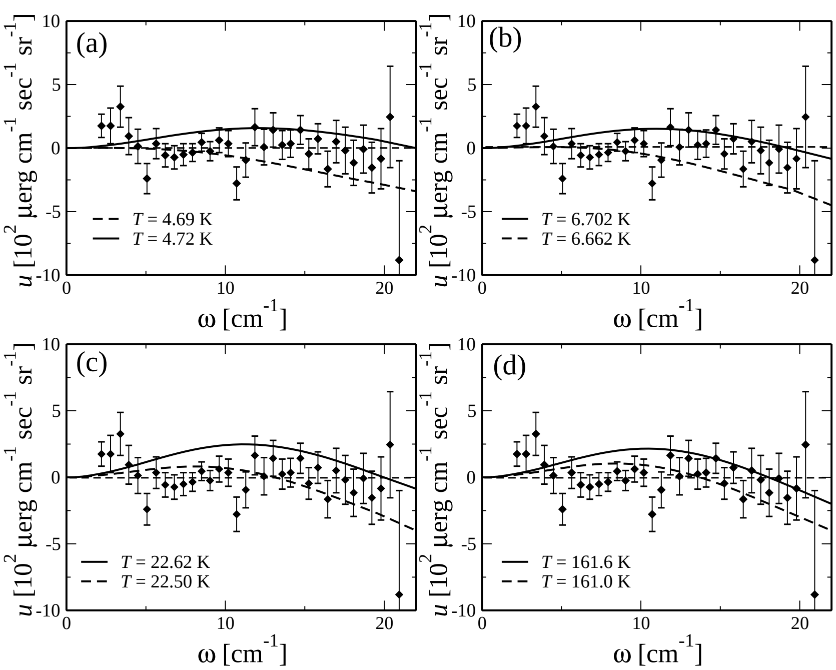

FIG. 6. We show the differential radiation between for the FIRAS spectrographer onboard COBE satellite. Superimposed with the experimental data the best fits for four different q values are also shown.

TABLES

Formalism	Result	
Stefan-Boltzmann' law	$U_1 = \sigma T^4$	$\sigma = (\pi^2 k^4 V)/(15\hbar^3 c^3)$
C-T Solution	$U_q^{CT} = 3\xi_3 kT\Gamma^{1-q}\left[\frac{2-q}{1-q}\right]\frac{\sum\limits_{m=0}^{\infty}\frac{\xi_3^m}{m!}\Gamma^{-1}\left[\frac{2-q}{1-q}+3m+3\right]}{\left[\sum\limits_{m=0}^{\infty}\frac{\xi_3^m}{m!}\Gamma^{-1}\left[\frac{2-q}{1-q}+3m\right]\right]^q}$	$\xi_3 = \frac{4\Gamma(3)\zeta(4)V}{\Gamma(3/2)}\left(\frac{\sqrt{\pi}kT}{hc(1-q)}\right)^3$
OLM Solution	$U_q^{OLM,R} = 3\xi_3 kT\frac{[1+(1-q)\beta U_q]^4}{(1-q)^4}\frac{\sum\limits_{m=0}^{\infty}B_m\Gamma^{-1}\left[\frac{1}{1-q}+3m+4\right]}{\sum\limits_{m=0}^{\infty}B_m\Gamma^{-1}\left[\frac{1}{1-q}+3m\right]}$	$B_m = \frac{\xi_3^m}{m!}\frac{[1+(1-q)\beta U_q]^{nm}}{(1-q)^{3n}}$
FA Aproximation	$U_q^{(FA)} \approx \sigma[1-(1-q)\theta]T^4$	$\theta = \zeta(5)\Gamma(6)/(2\zeta(4)\Gamma(4))$
OLM-FA Aproximation	$U_q^{NFA} \approx \sigma\frac{1-(1-q)\theta}{1+4(1-q)\sigma T^3/k}T^4$	
OLM Aproximation	$U_q^{OLM} \approx \sigma\frac{1}{1-4(1-q)\sigma T^3/k}T^4$	

TABLE I. The results of different generalizations of Stefan-Boltzmann's law.